Height fluctuations in interacting dimers

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Perfect matchings of $\mathbb{Z}^2$ and height function

Height function:

$$h(f') - h(f) = \sum_{b \in C_{f \rightarrow f'}} \sigma_b (1_{b \in M} - 1/4)$$

where $\sigma_b = +1$ if $b$ crossed with white on the right/left.
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Height function:

$$h(f') - h(f) = \sum_{b \in C_{f \rightarrow f'}} \sigma_b (1_{b \in M} - 1/4)$$

where $\sigma_b = +1/ -1$ if $b$ crossed with white on the right/left.

Crucial observation: white-to-black flux $(1_{b \in M} - 1/4)$ is divergence-free. Important point: $\mathbb{Z}^2$ is bipartite.
Non-interacting dimers (or uniform perfect matchings)

If $\Lambda$ is a large domain, e.g. the $2L \times 2L$ square/torus, many ($\approx \exp(cL^2)$) perfect matchings exist.

Call $\langle \cdot \rangle_{\Lambda;0}$ the uniform measure.
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Call $\langle \cdot \rangle_{\Lambda;0}$ the uniform measure.

Observe:

- By symmetry, on the torus, $\langle 1_{b \in M} \rangle_{\Lambda;0} = 1/4$ for every $b$, so that $\langle h(f) - h(f') \rangle_{\Lambda;0} = 0$.
- Dimers do not interact (except for hard-core constraint).
Non-interacting dimers (or uniform perfect matchings)

Known facts:

- Dimer-dimer correlations decay slowly:

\[ \lim_{\Lambda \rightarrow \mathbb{Z}^2} \langle 1_{b \in M} \; 1_{b' \in M} \rangle_{\Lambda,0} \approx |b - b'|^{-2} \]
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  \]

- Height fluctuations grow logarithmically:
  \[
  \lim_{\Lambda \to \mathbb{Z}^2} \text{Var}_{\Lambda,0}(h(f) - h(f')) \sim \frac{1}{\pi^2} \log |f - f'| \quad \text{as} \quad |f - f'| \to \infty
  \]

(see Kenyon-Okounkov-Sheffield for general bipartite graphs, periodic b.c.)
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- the height field is asymptotically Gaussian: for \( m \geq 3 \), the \( m \)th cumulant of \( h(f) - h(f') \) is

\[ \langle h(f) - h(f') ; m \rangle_{\Lambda,0} = o(\text{Var}_{\Lambda,0}(h(f) - h(f'))^{m/2}). \]

(recall: cumulants of \( X \) are zero for \( m \geq 3 \) iff \( X \) is Gaussian)
Interacting dimers

Associate an energy $\lambda \in \mathbb{R}$ to adjacent dimers:

I.e., with $N(M)$ the number of adjacent pairs of dimers in $M$,

$$\langle \cdot \rangle_{\Lambda,\lambda} = \frac{\sum_M e^{\lambda N(M)}}{Z_{\Lambda,\lambda}}.$$

Theorem [Giuliani, Mastropietro, T. 2014] If $|\lambda| \leq \lambda_0$ then:

- Fluctuations still grow logarithmically:

$$\lim_{\Lambda \to \mathbb{Z}^2} \text{Var}_{\Lambda,\lambda}(h(f) - h(f')) \sim \frac{K(\lambda)}{\pi^2} \log |f - f'|$$

with $K(\cdot)$ analytic and $K(0) = 1$;

- for $m \geq 3$, the $m^{th}$ cumulant of $h(f) - h(f')$ is bounded:

$$\sup_{f,f'} \lim_{\Lambda \to \mathbb{Z}^2} \langle h(f) - h(f'); m \rangle_{\Lambda,\lambda} \leq C(m).$$
Interacting dimers

- **Convergence to the GFF**
  If $|\lambda| \leq \lambda_0$ then convergence to Gaussian Free Field: if $\varphi \in C_0^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} \varphi(x)\,dx = 0$ then, as $\epsilon \to 0$,

  $$
  \epsilon^2 \sum_f \varphi(\epsilon f) h(f) \Rightarrow \int_{\mathbb{R}^2} \varphi(x) X(x) \,dx
  $$

  with $X$ the Gaussian Free Field of covariance

  $$
  - \frac{K(\lambda)}{2\pi^2} \log |x - y|.
  $$
Comments

- System remains “critical” even for $\lambda \neq 0$. 

$\lambda$ is a parameter in the system, and $\Lambda$ represents a suitable discretization of the domain $D \subset \mathbb{C}$.
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- Theorem proven with periodic boundary conditions.
- For $\lambda = 0$, Kenyon ’00 proves conformal invariance of height moments e.g.

$$g_D(x, y) = \lim_{L \to \infty} \langle (h_x - \langle h_x \rangle_{\Lambda, 0})(h_y - \langle h_y \rangle_{\Lambda, 0}) \rangle_{\Lambda, 0}$$

(lattice spacing $1/L$ tends to zero, $\Lambda$ is suitable discretization of domain $D \subset \mathbb{C}$ and $x, y$ tend to distinct points)

Challenge: proof for $\lambda \neq 0$
Non-interacting dimers: Kasteleyn theory

Partition functions and correlations given by determinants (or Pfaffians)

Define an antisymmetric $|\Lambda| \times |\Lambda|$ matrix $K$, indexed by lattice sites, as $K(x, x \pm e_1) = \pm 1$, $K(x, x \pm e_2) = \pm i$ and zero otherwise. Then,

$$Z = \sum_M 1 = Pf(K)$$

with, for antisymmetric $2n \times 2n$ matrix $A$,

$$Pf(A) = \frac{1}{2^n n!} \sum_{\pi} (-1)^\pi A_{\pi(1)} A_{\pi(2)} \cdots A_{\pi(2n-1)} A_{\pi(2n)}.$$
Non-interacting dimers: Kasteleyn theory

Similarly, if \( b_1 = (x_1, x_2) \), \( b_2 = (x_3, x_4) \) are two bonds \( (x_i \in \mathbb{Z}^2, |x_1 - x_2| = |x_3 - x_4| = 1) \), then

\[
\langle 1_{b_1 \in M} 1_{b_2 \in M} \rangle_{\wedge, 0} = K(b_1)K(b_2)\text{Pf}(M)
\]

with \( M \) the \( 4 \times 4 \) matrix with \( M_{ij} = K^{-1}(x_i, x_j) \).

E.g.

\[
\langle 1_{(x, x + e_1) \in M} 1_{(y, y + e_1) \in M} \rangle_{\wedge, 0} = K^{-1}(x, x + e_1)K^{-1}(y, y + e_1) - K^{-1}(x, y + e_1)K^{-1}(y, x + e_1)
\]
Inverse Kasteleyn matrix (or “propagator”)

The inverse matrix $K^{-1}$ can be computed explicitly, diagonalizing $K$:

$$
\lim_{\Lambda \to \mathbb{Z}^2} K^{-1}(x, y) = \int_{[-\pi, \pi]^2} \frac{dk}{(2\pi)^2} \frac{e^{-ik(x-y)}}{-i \sin k_1 + \sin k_2}
$$

Singularities at $(k_1, k_2) = (0, 0), (\pi, 0), (\pi, \pi), (0, \pi)$ produce $|x - y|^{-1}$ decay of $K^{-1}$:

$$
K^{-1}(x, 0) \xrightarrow{|x| \to \infty} \frac{1}{2\pi} \left[ \frac{1}{x_1 + ix_2} + \frac{(-1)^x_2}{x_1 - ix_2} \right]
$$
Back to height fluctuations (free case)

Recall $h(f') - h(f) = \sum_{b\in C_f \to f'} \sigma_b (1_{b\in M} - 1/4)$

One finds

$$\sigma_b \sigma_{b'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{b\in M}; 1_{b'\in M} \rangle_{\Lambda,0} = A_{b,b'} + B_{b,b'} + C_{b,b'}$$

$$= -\frac{1}{2\pi^2} \Re \left[ \Delta z_b \Delta z_{b'} \frac{1}{(z_b - z_{b'})^2} \right]$$

$$+ \text{Osc}(z_b, z_{b'}) \frac{1}{|z_b - z_{b'}|^2} + O(|z_b - z_{b'}|^{-3}).$$

Then [Kenyon-Okounkov-Sheffield ’06],

$$\sum_{b\in C_f \to f', b'\in C_{f'} \to f'} A_{b,b'} \sim -\frac{1}{2\pi^2} \Re \int_f^{f'} \frac{dz dz'}{(z - z')^2} = \frac{1}{\pi^2} \log |f - f'|.$$
Dimer-dimer correlations, interacting case

**Theorem** If \( \lambda \) is small, then

\[
\begin{align*}
\sigma_b \sigma_{b'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{b \in M}; 1_{b' \in M} \rangle_{\Lambda, \lambda} &= -\frac{K(\lambda)}{2\pi^2} \Re \left[ \Delta z_b \Delta z_{b'} \frac{1}{(z_b - z_{b'})^2} \right] \\
&+ \text{Osc}(z_b, z_{b'}) \frac{1}{|z_b - z_{b'}|^{2+\eta(\lambda)}} + O(|z_b - z_{b'}|^{-3+O(\lambda)}).
\end{align*}
\]

with \( K(\cdot), \eta(\cdot) \) analytic and \( K(0) = 1, \eta(0) = 0. \)
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\]

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Note:

- in the main term the critical exponent remains 2
- in the oscillating term it changes to \( 2 + \eta(\lambda) \) (non-universal).
Non-interacting dimers: “lattice free fermions”

Algebraic identity: Pfaffian can be written as “Grassmann Gaussian integral”

\[
\{\psi_x\}_{x \in \Lambda} \text{ Grassmann variables: } \psi_x \psi_y = -\psi_y \psi_x \text{ and } \psi_x^2 = 0.
\]
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All functions are polynomials: for instance,

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e^{\psi_x} = 1 + \psi_x + \psi_x^2/2 + ... = 1 + \psi_x
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\[ e^{\psi_x} = 1 + \psi_x + \frac{\psi_x^2}{2} + ... = 1 + \psi_x \]

Integration rules:

\[ \int \prod_{i=1}^{n} d\psi_i \, \psi_n \ldots \psi_1 = 1 \]

and

\[ \int \prod_{i=1}^{n} d\psi_i \, \psi_k \ldots \psi_1 = 0 \quad k < n. \]
Then,

\[ Pf(K) = \int \prod_{u \in \Lambda} d\psi_u e^{-\frac{1}{2} (\psi, K\psi)} \]

and

\[ K^{-1}(x, y) = \langle \psi_x \psi_y \rangle_0 := \frac{1}{Pf(K)} \int \prod_{u \in \Lambda} d\psi_u e^{-\frac{1}{2} (\psi, K\psi)} \psi_x \psi_y. \]

Also “fermionic Wick theorem”:

\[ \langle \psi_{x_1} \cdots \psi_{x_{2n}} \rangle_0 = \sum_{\text{pairings } \pi} \sigma_{\pi} \langle \psi_{x_{\pi(1)}} \psi_{x_{\pi(2)}} \rangle_0 \times \cdots \times \langle \psi_{x_{\pi(2n-1)}} \psi_{x_{\pi(2n)}} \rangle_0 \]
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“Fermions” because of anticommutation, “free” because exponential of quadratic form
Interacting dimers as interacting fermions

Similarly, the partition function of the interacting model is written as

\[ Z_{\Lambda, \lambda} = \frac{1}{Pf(K)} \int \prod d\psi_x \exp \left( -\frac{1}{2} (\psi, K \psi) + V(\psi) \right) = \left\langle \exp(V(\psi)) \right\rangle_{\Lambda, 0} \]

with

\[ V(\psi) = V_4(\psi) + V_6(\psi) + \ldots, \]

and

\[ V_4(\psi) = \lambda \sum_x \psi_x \psi_{x+e_1} \psi_{x+e_2} \psi_{x+e_1+e_2}, \]
Similarly, the partition function of the interacting model is written as

$$Z_{\Lambda, \lambda} = \frac{1}{Pf(K)} \int \prod d\psi_x \exp\left( -\frac{1}{2}(\psi, K\psi) + V(\psi) \right) \equiv \left\langle \exp(V(\psi)) \right\rangle_{\Lambda,0}$$

with

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and

$$V_4(\psi) = \lambda \sum_x \psi_x \psi_{x+e_1} \psi_{x+e_2} \psi_{x+e_1+e_2},$$

NB: for finite $\Lambda$, these are just exact identities, $V$ is a polynomial (finite degree).
Difficulties I: a combinatorial problem

Naif approach: perturbative expansion in $\lambda$

$$\langle \exp(V(\psi)) \rangle_{\Lambda,0} = \sum_n \frac{1}{n!} \langle V(\psi)^n \rangle_{\Lambda,0}.$$

Each expectation is computed via Wick’s rule as sum of “Feynman diagrams”. However, number of pairings is at least $(n!)^2$. Not summable.
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Solution: anticommutation rules $\Rightarrow$ relative signs $\Rightarrow$ gain a factor $n!$ (ideas form the ’80s, QFT; e.g. Gawedzki-Kupiaienen ’86,...).
Difficulties II: “infrared problem”

Due to slow decay of two-point function $K^{-1}$, many Fenyman diagrams are divergent (as $\Lambda \rightarrow \infty$).

A typical problem in Quantum Field Theory with massless fields.
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Constructive QFT (Benfatto, Brydges, Gallavotti, Gawedzki, Kupiainen, Rivasseau, Spencer...) provides the right tools to cure these problems.
Step 1: a change of variables

Decompose $K^{(-1)}(x, y) = \langle \psi_x \psi_y \rangle_0$ around the 4 singularities $p_1 = (0, 0), p_2 = (\pi, 0), p_3 = (\pi, \pi), p_4 = (0, \pi)$:

$$K^{-1}(x, y) = \sum_{\gamma=1}^{4} \int \frac{dk}{(2\pi)^2} \chi(k - p_\gamma) \frac{e^{-ik(x-y)}}{-i \sin k_1 + \sin k_2}$$  \hspace{1cm} (1)

i.e. rewrite

$$\psi_x = e^{ip_1 x} \psi_{x,1} + ie^{ip_2 x} \psi_{x,2} + ie^{ip_3 x} \psi_{x,3} + e^{ip_4 x} \psi_{x,4}$$

with

$$\langle \psi_x, \gamma \psi_y, \gamma' \rangle_0 = \delta_{\gamma, \gamma'} \int \frac{dk}{(2\pi)^2} \chi(k) \frac{e^{-ik(x-y)}}{-i \sin k_1 + (-1)^{\gamma+1} \sin k_2}$$

$$\sim \frac{\delta_{\gamma, \gamma'}}{4\pi} \frac{1}{(x_1 - y_1) + i(-1)^{\gamma+1}(x_2 - y_2)}$$
Step 1: a change of variables

This way, $V(\psi)$ becomes

$$V(\psi) = \lambda \sum \psi_{x,1} \psi_{x,2} \psi_{x,3} \psi_{x,4} + \text{higher order},$$
Step 2: multi-scale integration

- multiscale decomposition of the “free propagator” or of the field:
  \[ \psi_{x,\gamma} = \psi_{x,\gamma}^{(0)} + \psi_{x,\gamma}^{(-1)} + \psi_{x,\gamma}^{(-2)} + \cdots \]

  for each \( \psi_{x,\gamma}^{(h)} \), integration restricted to \( k \approx 2^h \);

- multiscale integration starting from short-distance scales: at each scale \( h \), effective potential

  \[ V^{(h)}(\psi_{\leq h}) = \lambda^{(h)} \sum_{x} \psi_{x,1}^{\leq h} \psi_{x,2}^{\leq h} \psi_{x,3}^{\leq h} \psi_{x,4}^{\leq h} + \text{higher order} \]

- flow equation for the effective coupling:
  \[ \lambda^{(h)} = \lambda^{(h+1)} + \beta(\lambda^{(h+1)}, \ldots, \lambda^{(0)}) \]

- key question: behavior of \( \lambda^{(h)} \) as \( h \to -\infty \).
Step 3: comparison with “relativistic model”

Important fact: the function $\beta(...)$ vanishes asymptotically for $h \to -\infty$, and $\lambda^{(h)} \to \lambda_{-\infty} = \lambda + O(\lambda^2)$. 
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Relies on works by Benfatto-Mastropietro on a related model (Thirring model) where, essentially, the denominator

$$-i \sin k_1 + (-1)^{j+1} \sin k_2$$

in $\langle \psi_x, \gamma \psi_y, \gamma \rangle_0$ is linearized and replaced by

$$-i k_1 + (-1)^{j+1} k_2.$$

Uniform smallness of $\lambda^{(h)}$ guarantees convergence of perturbation expansion.
Analogy with the 2D Ising model

Let $\mu_{\Lambda,0}$ be the Gibbs measure of the nearest-neighbor 2D Ising model at $T_c$, and $\mu_{\Lambda,\lambda}$ the one with Hamiltonian perturbed by $\lambda \sum_{x,y} \nu(x - y) \sigma_x \sigma_y$, at its critical point $T_c(\lambda)$. 

Greenblatt-Giuliani-Mastropietro '12: if $|\lambda| \leq \lambda_0$ and $\nu(\cdot)$ finite range, then (2) still true.
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- The analog of dimer-dimer correlations are energy-energy correlations: if $|x - x'| = |y - y'| = 1$

  $$\mu_{\Lambda,0}(\sigma_x \sigma_{x'}; \sigma_y \sigma_{y'}) \approx |x - y|^{-2}.$$  \hspace{1cm} (2)

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Analogy with the 2D Ising model

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- The analog of (square of) spin-spin correlations $\mu_{\Lambda,\lambda}(\sigma_x; \sigma_y)$ is the “electric correlator”

$$\mathcal{E}(f, f') = \langle e^{i\pi \alpha(h(f) - h(f'))} \rangle_{\Lambda,\lambda}, \quad \alpha = 1.$$  

Computation of $\mathcal{E}(f, f')$ is hard even for $\lambda = 0$, (Pinson ’04, Dubedat ’11).
Conclusions

• Novelties:
  • match between constructive QFT methods (huge literature) and some (simple) discrete complex analysis ideas
  • control of a non-local fermionic observable (height field) in a non-integrable case

While critical exponent of dimer-dimer correlations is not universal, large-scale GFF behavior is;

To be done (major difficulties):
  • get rid of periodic b.c., work with general domains (necessary to study e.g. conformal invariance).
  • control the exponential of the height function. Our result suggests:
    \[ E(f, f') \approx |f - f'| - \alpha 2^{K(\lambda)} / 2 \]
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  - get rid of periodic b.c., work with general domains (necessary to study e.g. conformal invariance).
  - control the exponential of the height function. Our result suggests:

\[ \mathcal{E}(f, f') \approx |f - f'|^{-\alpha^2 K(\lambda)/2} \]